Unsteady interaction of a shock wave with a cellular vortex field

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The transient effects generated when a shock wave is suddenly disturbed by a field of cellular vortices have been studied. Both the pressure disturbance on the shock and the local shock velocity are found to be strong functions of the cell geometry. Disturbances are resolved into transient components and sinusoidal components of constant amplitude. The transients are found to die out as $t^{-3/2}$, t being the interaction time, except for one particular case of the cell geometry for which they diminish as $t^{-1/2}$. Furthermore, the analysis indicates that the initial magnitude of the transient components may be quite appreciable in comparison with the sinusoidal component. The theory is extended to treat the convection through the shock of a single column of vortex cells.

1. Introduction

It has been demonstrated (Kovasznay 1953, Chu & Kovasznay 1958) that an arbitrary weak disturbance in a uniform compressible flow can be decomposed into component modes of vorticity, sound and entropy. As a consequence of linearization these fields have the important property of being independent of each other in the absence of natural boundaries. Across a shock-wave, however, the presence of large gradients in the basic flow make it necessary to retain the non-linearity of the fundamental equations. This results in a coupling of modes. In particular a first-order vorticity fluctuation, upon passing through a shock front, generates sound and entropy fields behind the shock (Ribner 1953). At the same time, local displacement of the shock occurs, which means that the analytical problem is of the nature of an unsteady boundary value problem with a free boundary.

The specific model employed here is illustrated in figure 1. A two-dimensional cellular vortex field with a discrete front is convected into an initially plane, normal shock-wave. The objective of the analysis is to investigate the transient behaviour of the shock front and the pressure disturbance generated on the downstream face of the shock.

Interaction studies of a similar nature have been carried out by Ribner (1953), Moore (1953) and Chang (1957). These differ from the present analysis in that they treat the case of a single harmonic disturbance of infinite extent in space. Such a model can be reduced to a steady state problem by a simple translation of the co-ordinate system. It therefore lacks the transient or 'initial' behaviour of the present model. On the other hand, the treatment by Ram & Ribner (1957) of shock interaction with a single vortex exhibits transient behaviour only. The model considered here occupies a position between the two in that it is comprised of transients which subside, leaving a completely periodic behaviour which may be compared with the results of Ribner (1953), Moore (1953) and Chang (1957). That transient behaviour is significant is borne out by the analysis which indicates that transient components of pressure may in some cases be an order of magnitude larger than the eventual steady-state amplitude. Thus, transients should be considered in any estimates of disturbances generated by an unsteady interaction process.

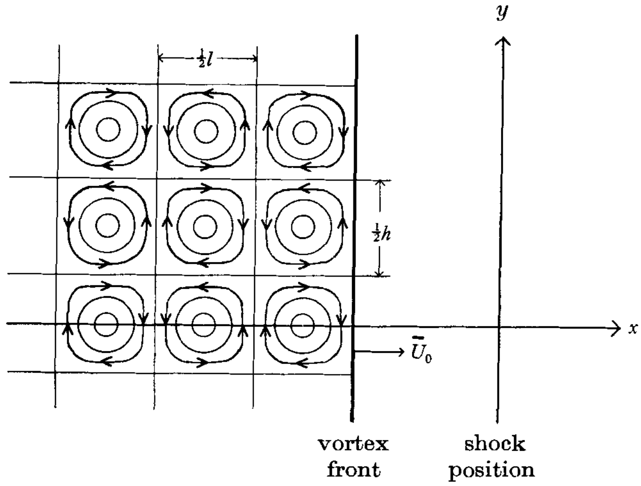

FIGURE 1. Cellular vortex field.

2. Fundamental equations

The flow field is described in co-ordinates fixed at the mean position of the shock as indicated in figure 1. To represent fluctuations, $\delta(\ )$, of pressure P, density ρ_T, velocity $\mathbf{V}$ and entropy Σ the following non-dimensional variables are employed:

$$p = \delta P/\gamma \bar{P},$$
$$\rho = \delta \rho_T/\bar{\rho},$$
$$\sigma = \delta\Sigma/C_P,$$
$$\mathbf{v} = \mathbf{i}u + \mathbf{j}v = \mathbf{i}\frac{\delta U}{a} + \mathbf{j}\frac{\delta V}{a} = \frac{\delta \mathbf{V}}{a},$$

where a is the speed of sound in the basic flow, C_p is the specific heat at constant pressure, and γ is the ratio of specific heats. Basic flow quantities are denoted by $(\bar{\ })$. In terms of these variables the linearized equations of motion for the region behind the shock are, denoting this region by $(\ )_1$,

$$\left.\begin{aligned} &\text{continuity,} && \frac{Dp_1}{D\tau} + \nabla . \mathbf{v}_1 = 0; \\ &\text{momentum,} && \frac{D\mathbf{v}_1}{D\tau} + \nabla p_1 = 0; \\ &\text{energy,} && \frac{D\sigma_1}{D\tau} = 0; \\ &\text{state,} && p_1 - \rho_1 = \sigma_1. \end{aligned}\right\} \tag{2.1}$$

Here $\tau = a_1 t$ and $D/D\tau$ is the linearized Stokes derivative:

$$\frac{D}{D\tau} = \frac{\partial}{\partial\tau} + M_1 \frac{\partial}{\partial x},$$

where
$$M_1 = \overline{U}_1/a_1.$$

3. Specification of upstream disturbance

Denoting the region ahead of the shock by $(\ )_0$, the following velocity perturbation in the oncoming flow is assumed:

$$\left.\begin{aligned} &\left.\begin{aligned} u_0(x,y,t) &= |u_0| \sin\kappa(\overline{U}_0 t - x)\sin\lambda y,\\ v_0(x,y,t) &= -\frac{\kappa}{\lambda}|u_0|\cos\kappa(\overline{U}_0 t - x)\cos\lambda y,\end{aligned}\right\} \quad (\overline{U}_0 t - x > 0);\\ &\left.\begin{aligned} u_0(x,y,t) &= 0,\\ v_0(x,y,t) &= 0,\end{aligned}\right\} \quad (\overline{U}_0 t - x \leqslant 0).\end{aligned}\right\} \tag{3.1}$$

The field represented by this perturbation has the cellular structure illustrated in figure 1, which shows the streamlines as they would appear to an observer moving with the basic flow. In the interior of the cellular pattern it can be verified, from the linearized equations of motion, that to the first order

$$p_0 = \rho_0 = \sigma_0 = 0. \tag{3.2}$$

The region near the leading edge of the vortex field requires special consideration. The pressure on the upstream face of the leading edge is second-order in magnitude. On the downstream face the pressure is identically zero. To maintain this pressure unbalanced a thin plate is imagined held at the leading edge of the vortex field until the time $t = 0$, at which instant it is suddenly removed. Immediately after withdrawal of the plate the vortex field is convected into the shock.

Once the vortex front passes through the shock no further flow modification generated at the front can propagate upstream ahead of the shock. This means that the presence of a vortex front no longer affects the disturbance being convected into the shock from upstream. But at the instant the plate is removed the vortex front is just ahead of the shock. It therefore remains to consider the magnitude of the initial disturbance generated during the infinitesimal time interval before the vortices pass through the shock. That the flow modifications generated in this interval may be neglected can be shown by the following considerations. Locally, the instantaneous removal of the plate and the resulting pressure unbalance are roughly equivalent to the diaphragm shattering in a shock tube. If the pressure difference across such a diaphragm is represented by the non-dimensionalized pressure term $p = \delta P/\gamma P$, it can readily be shown by linearization of the Rankine–Hugoniot equations that, for small values of p, the flow velocity behind the resulting weak shock would be, to the first approximation

$$u = \tfrac{1}{2}p. \tag{3.3}$$

This indicates that the velocity field generated by removing the plate is initially of the same order of magnitude as the pressure unbalance. Since this latter term is of second order in magnitude, the initial velocity disturbance is also and may be neglected. The disturbance specified by equations (3.1) and (3.2) is thus correct to the first order.

4. Boundary conditions

Equations (2.1) supply the differential relations satisfied at a point downstream of the shock. To complete the formulation, boundary conditions appropriate to these equations are required. These are provided by the values of p_+, ρ_+, v_+ and σ_+, on the downstream face of the shock, which are in turn related to p_-, ρ_-, v_- and σ_-, on the upstream face, through the Rankine–Hugoniot equations. To arrive at the specific relationships a local element of the shock front is isolated as shown in figure 2. Its displacement from its undisturbed position is denoted by $\psi(y,\tau)$ and its velocity with respect to the upstream flow field by $\mathbf{V}_- + \mathbf{i}a_1\psi_\tau$. The angle between the shock and the y-axis is assumed small and equal to ψ_y. These quantities are substituted into the Rankine–Hugoniot equations which are then linearized. The result is a set of simultaneous equations for $p_+, \rho_+, v_+, \sigma_+$ in terms of $p_-, \rho_-, v_-, \sigma_-$.

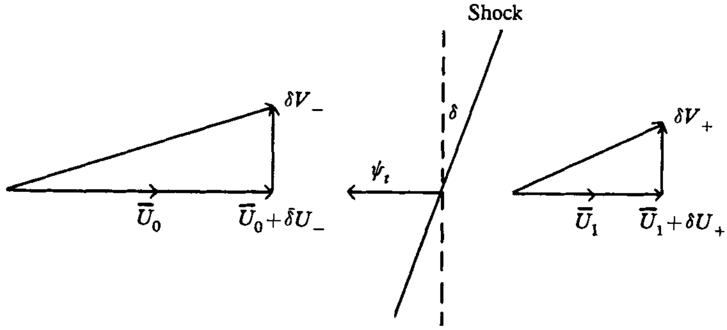

FIGURE 2. Local flow conditions at shock.

The procedure has been carried out by Chang (1957) for the general case of interactions with an oblique shock. Specializing Chang's results for the present case the following equations hold at the shock front:

$$\sigma_+ = \Omega_{13}u_- + \Pi_{11}\psi_\tau, \tag{4.1 A}$$

$$p_+ = \Omega_{23}u_- + \Pi_{21}\psi_\tau, \tag{4.1 B}$$

$$u_+ = \Omega_{33}u_- + \Pi_{31}\psi_\tau, \tag{4.1 C}$$

$$v_+ = \Omega_{44}u_- + \Pi_{41}\psi_y, \tag{4.1 D}$$

where Ω_{ij} and Π_{ij}, as obtained from Chang (1957), are

$$\Omega_{13} = (\gamma-1)(1-\bar{\rho}_1/\bar{\rho}_0)^2(M_1/M_0)^2,$$

$$\Omega_{23} = -M_1(1-\rho_1/\rho_0)[2+(\gamma-1)(1-\bar{\rho}_1/\bar{\rho}_0)M_1^2](M_1/M_0)/(1-M_1^2),$$

$$\Omega_{33} = (M_1/M_0)[1-(M_1\bar{\rho}_1/\bar{\rho}_0)^2+\gamma(1-\bar{\rho}_1/\bar{\rho}_0)^2M_1^2]/(1-M_1^2),$$

$$\Omega_{44} = \bar{\rho}_1M_1/\bar{\rho}_0M_0,$$

$$\Pi_{11} = -(\gamma-1)(1-\bar{\rho}_0/\bar{\rho}_1)^2(\bar{\rho}_1/\bar{\rho}_0)M_1,$$
$$\Pi_{21} = -M_1(1-\bar{\rho}_0/\bar{\rho}_1)[2+(\gamma-1)(1-\bar{\rho}_1/\bar{\rho}_0)M_1^2]/(1-M_1^2),$$
$$\Pi_{31} = (1-\bar{\rho}_0/\bar{\rho}_1)[1+M_1^2+(\gamma-1)(1-\bar{\rho}_1/\bar{\rho}_0)M_1^2]/(1-M_1^2),$$
$$\Pi_{41} = -(1-\bar{\rho}_1/\bar{\rho}_0)M_1.$$

Along with the specified velocity perturbations at the shock, these equations also include derivatives of the shock displacement. Since displacement is determined only after solution of the flow problem, the shock constitutes a free boundary of the downstream region.

5. The general solution in the region behind the shock

To isolate the pressure p_1 the divergence of the momentum equation, (2.1), is formed and div $\mathbf{v}_1$ eliminated by introducing the continuity relation. The expanded result is

$$\frac{\partial^2 p_1}{\partial\tau^2}+2M_1\frac{\partial^2 p_1}{\partial x\,\partial\tau}-(1-M_1^2)\frac{\partial^2 p_1}{\partial x^2}-\frac{\partial^2 p_1}{\partial y^2}=0. \tag{5.1}$$

In view of the particular time-dependence of the disturbance a solution will be sought by the method of Laplace-transformation. Introducing the pressure transform

$$\tilde{p}_1(x,y,s) = L[p_1(x,y,\tau)] = \int_0^\infty p_1(x,y,\tau)\,e^{-s\tau}d\tau, \tag{5.2}$$

the transform of equation (5.1) is

$$s^2\tilde{p}_1+2M_1s\frac{\partial\tilde{p}_1}{\partial x}-(1-M_1^2)\frac{\partial^2\tilde{p}_1}{\partial x^2}-\frac{\partial^2 p_1}{\partial y^2}=0. \tag{5.3}$$

A general solution of equation (5.3) is obtainable by separation of variables, giving

$$\tilde{p}_1(s) = (A_1\sin\lambda y+A_2\cos\lambda y)$$
$$\times(B_1\exp[\{M_1s-(s^2+\lambda^2\mu^2)^{\frac{1}{2}}\}x/\mu^2]+B_2\exp[\{M_1s+(s^2+\lambda^2\mu^2)^{\frac{1}{2}}\}x/\mu^2]), \tag{5.4}$$

where
$$\mu^2 = 1-M_1^2.$$

6. Application of boundary conditions

To satisfy the condition that no disturbances originate in the downstream region we must have

$$B_2 = 0. \tag{6.1}$$

Additional boundary conditions are furnished by equations (4.1 A–D) which must be satisfied on the shock front. The displacement derivatives ψ_τ, ψ_y are first eliminated by cross-differentiation. Then, with the aid of the fundamental equations (2.1) to eliminate derivatives of u_+ and v_+, the variable p_+ is isolated from the remaining equations to obtain a differential expression for p_+ which must be satisfied by the solution for p_1. This is

$$(1-AM_1)\frac{\partial^2 p_+}{\partial\tau^2}+A(1-M_1^2)\left(\frac{\partial^2 p_1}{\partial x\,\partial\tau}\right)_{x=0}-\frac{AM_1}{C}\frac{\partial^2 p_+}{\partial\tau^2}$$
$$=\frac{AM_1}{C}\left[D\frac{\partial^2 u_-}{\partial y^2}+E\frac{\partial^2 v_-}{\partial\tau\,\partial y}\right]+B\frac{\partial^2 u_-}{\partial\tau^2}$$
$$=\phi(\tau)\sin\lambda y, \tag{6.2}$$

where $$A = \Pi_{21}/\Pi_{31}, \quad B = \Omega_{23} - A\Omega_{33}, \quad C = \Pi_{21}/\Pi_{41},$$
$$D = \Omega_{23}, \quad E = -\Pi_{21}\Omega_{44}/\Pi_{41}.$$

$\phi(\tau)$ is obtained by direct substitution of $u_- = u_0(0, y, \tau)$ etc., into the right-hand side of (6.1), giving

$$\phi(\tau) = |u_0| [\phi_0 \sin(\kappa\tau\bar{U}_0/a_1) + \phi_1 \delta(\kappa\tau\bar{U}_0/a_1)], \tag{6.3}$$

where $$\phi_0 = (AM_1/C)(D\lambda^2 + E\kappa_1^2\bar{U}_0/a_1) - B\kappa^2(\bar{U}_0/a_1)^2,$$
$$\phi_1 = AM_1 D\lambda^2/C - \phi_0,$$

and $\delta(\kappa\tau\bar{U}_0/a_1)$ represents the Dirac delta function arising from discontinuities in the derivatives of u_- and v_-. The absence of cosine terms in equation (6.2) immediately allows us to set $A_2 = 0$. The solution must then be of the form

$$\tilde{p}_1(s) = F(s)\exp[\{M_1 s - (s^2 + \lambda^2\mu^2)^{\frac{1}{2}}\} x/\mu^2] \sin\lambda y, \tag{6.4 A}$$

or in its inverted form

$$p_1(x, y, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp[\{M_1 s - (s^2 + \lambda^2\mu^2)^{\frac{1}{2}}\}(x/\mu^2) + s\tau]\, ds \sin\lambda y. \tag{6.4 B}$$

Substituting equation (6.4 B) into the boundary condition—equation (6.2)—and setting $x = 0$ an equation is obtained for $F(s)$ which may be put into the form

$$F(s) = \tilde{\phi}(s)\frac{(1+A)s^2 + (AM_1/C)\lambda^2 - AS\{s - (s^2 + \lambda^2\mu^2)^{\frac{1}{2}}\}}{(1-A^2)s^4 + \{2(AM_1/C) - \mu^2\}\lambda^2 s^2 + (AM_1/C)^2\lambda^4}. \tag{6.5}$$

From equation (6.4 B) it is evident that $F(s)$ is the transform of $p_+(y\tau)$.

7. Inversion of $F(s)$

With the aid of standard tables and the convolution theorem the inversion of $F(s)$ is achieved in a straightforward manner. The resulting expression however is long and unwieldy. A considerable reduction in the size and number of terms is obtained if we note that over a range of Mach number up to about six, the parameter A is very nearly equal to -1. If this approximation is introduced, equation (6.5) reduces to the simpler form

$$F(s) = -\frac{\mu^2 C}{2M_1 + C\mu^2}\left[\frac{s - (s^2 + \lambda^2\mu^2)^{\frac{1}{2}}}{\lambda\mu}\right]\left[\frac{s}{s^2 + \beta^2\lambda^2}\right]\frac{\tilde{\phi}(s)}{\lambda\mu} + \left[\frac{M_1}{2M_1 + C\mu^2}\right]\frac{\tilde{\phi}(s)}{s^2 + \beta^2\lambda^2}, \tag{7.1}$$

where $\beta^2 = -M_1^2/2C(2M_1 + C\mu^2)$, $\sqrt{\beta^2}$ being positive. Finally, introducing the parameter $\alpha = \bar{U}_0/a_1$ and the explicit form of $\tilde{\phi}(s)$, the inversion of $F(s)$ is found, yielding

$$\frac{p_+}{|u_0|\sin\lambda y} = C_\beta\left[I_\beta(\tau) + \frac{M_1}{C\beta\mu}\sin(\beta\lambda\tau)\right] + C_\alpha\left[I_\alpha(\tau) + \frac{\lambda\mu}{\alpha\kappa}\frac{M_1}{C\mu^2}\sin(\alpha\kappa\tau)\right], \tag{7.2}$$

where

$$I_\beta(\tau) = \int_0^\tau \cos[\beta\lambda(\tau - \tau')]\frac{J_1(\lambda\mu\tau')}{\tau'}\, d\tau',$$

$$I_\alpha(\tau) = \int_0^\tau \cos[\alpha\kappa(\tau - \tau')]\frac{J_1(\lambda\mu\tau')}{\tau'}\, d\tau',$$

$$C_\beta = \frac{\mu^2 C}{2M_1 + C\mu^2}\frac{\alpha\kappa}{\lambda\mu}\left[\frac{\phi_0}{\alpha^2\kappa^2 - \beta^2\lambda^2} + \frac{\phi_1}{\alpha^2\kappa^2}\right], \quad C_\alpha = -\frac{\mu^2 C}{2M_1 + C\mu^2}\frac{\alpha\kappa}{\lambda\mu}\left[\frac{\phi_0}{\alpha^2\kappa^2 - \lambda^2\beta^2}\right].$$

8. Solution for large values of $\lambda\mu\tau$

The asymptotic behaviour of I_α and I_β for $\lambda\mu\tau \gg 1$ may be determined by the method of steepest descents (Jeffreys & Jeffreys 1950). For a line integral in the complex domain of the form

$$I = \int_R \chi(s)\, e^{f(s)} ds, \tag{8.1}$$

where $f(s)$ is analytic with a large positive real part, and R is the path of steepest descent of $f(s)$, a first approximation is given by

$$I = \sum_{\substack{\text{all saddle}\\ \text{points } s_0}} \frac{(2\pi)^{\frac{1}{2}}\, \chi(s_0)\, e^{f(s_0)}\, e^{i\xi}}{|f''(s_0)|^{\frac{1}{2}}}, \tag{8.2}$$

s_0 being the location of a saddle point of $f(s)$ and ξ the angle with the positive real axis made by R as it passes through s_0.

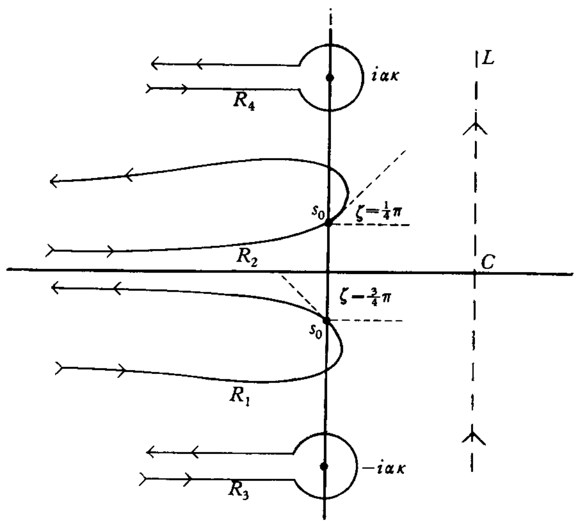

FIGURE 3. Path of steepest descent for evaluation of I_α for $\lambda\mu\tau \gg 1$.

The Laplace representation of I_α may be written in the form of equation (8.1) as follows

$$I_\alpha = \frac{1}{2\pi i}\int_L \frac{s}{s^2+\alpha^2\kappa^2} \exp\left[s\tau - \ln\left\{\frac{s+(s^2+\lambda^2\mu^2)^{\frac{1}{2}}}{\lambda\mu}\right\}\right] ds, \tag{8.3}$$

L denoting a line from $C-i\infty$ to $C+i\infty$ with $C > Im[\alpha]$. As shown in figure 3 L may be deformed so that part of it coincides with the paths R_1, R_2 of steepest descent for which equation (8.2) holds. Integration over the remaining parts R_3, R_4 may be carried out exactly by residue theory. The result, valid for large $\lambda\mu\tau$ and $\alpha\kappa/\lambda\mu \neq 1$, is:

$$I_\alpha = \frac{(2/\pi)^{\frac{1}{2}}}{(\alpha\kappa/\lambda\mu)^2-1}\, \frac{\cos\left[\lambda\mu\tau - \cos^{-1}(1/\lambda\mu\tau)+\frac{1}{4}\pi\right]}{(\lambda\mu\tau)^{\frac{3}{2}}} + \nu_1 \sin(\kappa\overline{U}_0 t) + \nu_2 \cos(\kappa\overline{U}_0 t), \tag{8.4}$$

where
$$\left.\begin{aligned}\nu_1 &= (\alpha\kappa/\lambda\mu)\,[1-\{1-(\lambda\mu/\alpha\kappa)^2\}^{\frac{1}{2}}],\\ \nu_2 &= 0,\end{aligned}\right\}\quad \left(\frac{\alpha\kappa}{\lambda\mu}>1\right);$$
$$\left.\begin{aligned}\nu_1 &= \alpha\kappa/\lambda\mu,\\ \nu_2 &= \{1-(\alpha\kappa/\lambda\mu)^2\}^{\frac{1}{2}},\end{aligned}\right\}\quad \left(\frac{\alpha\kappa}{\lambda\mu}<1\right).$$

A similar expression holds for I_β except that β/μ is always greater than unity. Substituting these results into equations (7.2) we obtain the result for $\alpha\kappa/\lambda\mu \neq 1$,

$$\begin{aligned}\frac{[p_+(y,\tau)]_{\tau\to\infty}}{|u_0|\sin\lambda y} &= \sqrt{\frac{2}{\pi}}\left[\frac{C_\beta\mu^2}{\beta^2-\mu^2}+\frac{C_\alpha}{(\alpha\kappa/\lambda\mu)^2-1}\right]\frac{\cos[\lambda\mu\tau-\cos^{-1}(1/\lambda\mu\tau)+\frac{1}{4}\pi]}{(\lambda\mu\tau)^{\frac{3}{2}}}\\ &\quad + C_\alpha\left\{\left[\nu_1+\frac{M_1}{C\mu^2}\frac{\lambda\mu}{\alpha\kappa}\right]\sin(\kappa\bar{U}_0t)+\nu^2\cos(\kappa\bar{U}_0t)\right\}\\ &= \frac{p_{+\mathrm{trans}}}{|u_0|\sin\lambda y}+\frac{p_{+\infty}}{|u_0|\sin\lambda y}. \end{aligned} \tag{8.5}$$

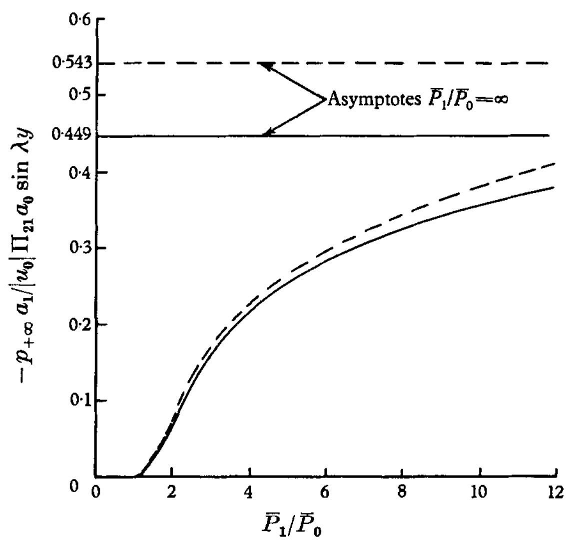

FIGURE 4. Comparison of present theory with Ribner (1953) for determination of amplitude of $p_{+\infty}$, showing $-p_{+\infty}a_1/|u_0|\,\Pi_{21}\,a_0\sin\lambda y$ *vs* shock strength $\bar{P}_1/\bar{P}_0$, for $\lambda=\kappa$. ———, Werner ($A=-1$); – – –, Ribner.

For $\alpha\kappa/\lambda\mu = 1$ the method of steepest descents cannot be applied to I_α. However, it will be shown in § 9 that for this case I_α can be expressed in the simple form:

$$I_\alpha = -J_1(\lambda\mu\tau)+\sin(\kappa\bar{U}_0t),\quad (\alpha\kappa/\lambda\mu=1). \tag{8.6}$$

Thus, for $\alpha\kappa/\lambda\mu = 1$,

$$\begin{aligned}[p_+(y,\tau)]_{\tau\to\infty} = \Bigg\{C_\alpha J_1(\lambda\mu\tau)+\sqrt{\frac{2}{\pi}}\left[\frac{C_\beta\mu^2}{\beta^2-\mu^2}\right]\frac{\cos[\lambda\mu\tau-\cos^{-1}(1/\lambda\mu\tau)+\frac{1}{4}\pi]}{(\lambda\mu\tau)^{\frac{3}{2}}}\\ +C_\alpha\left[1+\frac{M_1}{C\mu^2}\right]\sin(\kappa\bar{U}_0t)\Bigg\}.\end{aligned} \tag{8.7}$$

As τ approaches ∞, $p_{+\mathrm{trans}}$ dies out as $(\lambda\mu\tau)^{-\frac{3}{2}}$ for $\alpha\kappa/\lambda\mu \neq 1$, and as $(\lambda\mu\tau)^{-\frac{1}{2}}$ for $\alpha\kappa/\lambda\mu = 1$. The remaining term $p_{+\infty}$ varies sinusoidally in time with a fixed amplitude and a phase which depends on the parameter $\alpha\kappa/\lambda\mu$. The value of $p_{+\infty}$ should be predicted by the quasi-steady theories of Ribner (1953) and Chang (1957). Accordingly, Ribner's theory applied to two vorticity waves oriented at $\pm 45°$ to the mean flow direction is compared with the present theory for the equivalent case of $\kappa = \lambda$ (square vortex cells). For this configuration $\alpha\kappa/\lambda\mu > 1$. In figure 4, the amplitude of $p_{+\infty}a_1/|u_0|\Pi_{21}a_0$ is given as a function of shock strength P_1/P_0 for both Ribner's theory and the present determination. It is evident from this comparison that the approximation $A = -1$ (which aside from the linearization of the problem is the only one made) is applicable even for shock strengths as high as ten.

9. Solutions for small values of $\lambda\mu\tau$

To complete the analysis a representation is sought which describes $p_+(y,\tau)$ near the origin but which can be extended to values of τ for which the asymptotic solutions first become valid.

The case $\alpha\kappa/\lambda\mu > 1$

From the asymptotic results for I_α and I_β it is known that I_α, I_β consist of a transient term which approaches zero as $\tau \to \infty$ and a sine term of fixed amplitude. If two new functions $I_{\alpha t}$, $I_{\beta t}$ are defined by subtracting out the sinusoidal components of I_α and I_β, i.e.

$$I_{\alpha t} = I_\alpha - \frac{\alpha\kappa}{\lambda\mu}\left[1-\left\{1-\left(\frac{\lambda\mu}{\alpha\kappa}\right)^2\right\}^{\frac{1}{2}}\right]\sin(\alpha\kappa\tau) \tag{9.1 A}$$

and

$$I_{\beta t} = I_\beta - \frac{\beta}{\mu}\left[1-\left\{1-\left(\frac{\mu}{\beta}\right)^2\right\}^{\frac{1}{2}}\right]\sin(\beta\lambda\tau), \tag{9.1 B}$$

then $I_{\alpha t}$, $I_{\beta t}$ are damped oscillatory functions approaching zero as $\tau \to \infty$. For such functions a method of expansion due to Cambi (1956) is particularly well-suited.

A new transform function is defined as follows:

$$\Lambda[f(\tau)] = \lambda\mu\cosh q\int_0^\infty e^{-\tau\lambda\mu\sinh q}f(\tau)\,d\tau. \tag{9.2}$$

If $f(\tau)$ is equal to the nth-order Bessel function $J_n(\lambda\mu\tau)$, we obtain the simple relation

$$\Lambda[J_n(\lambda\mu\tau)] = e^{-nq}. \tag{9.3}$$

This result is applied by first obtaining the Λ transform of $I_{\alpha t}$

$$\begin{aligned}\Lambda[I_{\alpha t}] &= \lambda\mu\cosh q\bar{I}_{\alpha t}(\lambda\mu\sinh q)\\ &= -\frac{b_\alpha e^{4q}+(b_\alpha+1)e^{2q}+1}{e^{5q}+d_\alpha e^{3q}+e^q},\end{aligned} \tag{9.4}$$

where $$b_\alpha = 2\left(\frac{\alpha\kappa}{\lambda\mu}\right)^2\left[1-\left\{1-\left(\frac{\lambda\mu}{\alpha\kappa}\right)^2\right\}^{\frac{1}{2}}\right]-1,\quad d_\alpha = \left(2\frac{\alpha\kappa}{\lambda\mu}\right)^2-2.$$

Dividing the denominator of equation (9·4) directly into the numerator, $\Lambda[I_{\alpha t}]$ may be expressed as a series in e^{-nq},

$$\Lambda[I_{\alpha t}] = -\sum_{n\,\text{odd}} f_{\alpha n}e^{-nq}, \tag{9.5}$$

the inversion of which follows from equation (9·3),

$$I_{\alpha t} = -\sum_{n\ \mathrm{odd}} f_{\alpha n} J_n(\lambda\mu\tau), \tag{9.6}$$

where

$$\begin{aligned} f_{\alpha 1} &= b_\alpha, \\ f_{\alpha 3} &= -(d_\alpha f_{\alpha 1} - b_\alpha - 1), \\ f_{\alpha 5} &= -(d_\alpha f_{\alpha 3} + f_{\alpha 1} - 1), \\ f_{\alpha n} &= -(d_\alpha f_{\alpha(n-2)} + f_{\alpha(n-4)}) \quad (n \geq 7), \end{aligned}$$

with corresponding expressions holding for $I_{\beta t}$.

The reason for the particular choice of $I_{\alpha t}$ now becomes more evident. In working with an expansion in $J_n(\lambda\mu\tau)$ it was desired to apply this expansion to a function whose behaviour was not very different from the Bessel function (both $I_{\alpha t}$ and $J_n(\lambda\mu\tau)$ are damped oscillatory functions).

The case $\alpha\kappa/\lambda\mu < 1$

For this case we have found that $[I_\alpha]_{\tau\to\infty}$ approaches a sine wave of unit amplitude but shifted in phase from the original disturbance. For this case then, we construct $I_{\alpha t}$ by subtracting out only the sine component, giving

$$I_{\alpha t} = I_\alpha - \frac{\alpha\kappa}{\lambda\mu}\sin(\alpha\kappa\tau), \tag{9.7}$$

taking note of the fact that now

$$[I_{\alpha t}]_{\tau\to\infty} = \left\{1 - \left(\frac{\alpha\kappa}{\lambda\mu}\right)^2\right\}^{\frac{1}{2}} \cos(\alpha\kappa\tau). \tag{9.8}$$

Repetition of the procedure for $\alpha\kappa/\lambda\mu > 1$ yields an expression for $I_{\alpha t}$ similar to equation (9.6) with

$$b_\alpha = 2\left(\frac{\alpha\kappa}{\lambda\mu}\right)^2 - 1, \quad d_\alpha = 2\left(\frac{\alpha\kappa}{\lambda\mu}\right)^2 - 2.$$

$I_{\beta t}$, of course, remains unchanged since $\beta > \mu$ for all shock strengths. Although $I_{\alpha t}$ no longer goes to zero as before, d_α is sufficiently small to ensure good convergence properties for the expansion.

The case $\alpha\kappa/\lambda\mu = 1$

This case is contained in both the previous cases and we have

$$b_\alpha = 1, \quad d_\alpha = 1, \quad f_{\alpha 1} = 0, \quad f_{\alpha n} = 0 \quad (n > 1).$$

$I_{\alpha t}$ is then simply

$$I_{\alpha t} = -J_1(\lambda\mu\tau), \tag{9.9}$$

and I_α reduces to the form given in equation (8.6).

In terms of $I_{\alpha t}$, $I_{\beta t}$ the pressure disturbance is given by

$$\begin{aligned} \frac{p_+}{|u_0|\sin\lambda y} &= C_\beta I_{\beta t} + C_\alpha[I_{\alpha t} - \nu_2\cos(\kappa\bar{U}_0 t)] + \frac{p_{+\infty}}{|u_0|\sin\lambda y} \\ &= \frac{p_{+\mathrm{trans}}}{|u_0|\sin\lambda y} + \frac{p_{+\infty}}{|u_0|\sin\lambda y}. \end{aligned} \tag{9.10}$$

10. Local shock velocity perturbation

The non-dimensional shock velocity $\psi_t/|\delta U_0|$ may be obtained from the boundary condition equation (4.1 B) written in the form

$$\frac{\psi_t}{|\delta U_0|} = \frac{\delta U_-}{|\delta U_0|} + \frac{p_+}{|u_0|\Pi_{21}a_0/a_1}. \tag{10.1}$$

Substitution of δU_- and p_+ results in the expression

$$\frac{\psi_t}{|\delta U_0|\sin\lambda y} = G\sin(\kappa\overline{U}_0 t+\theta) + \frac{p_{+\text{trans}}}{|u_0|\Pi_{21}(a_0/a_1)\sin\lambda y}, \tag{10.2}$$

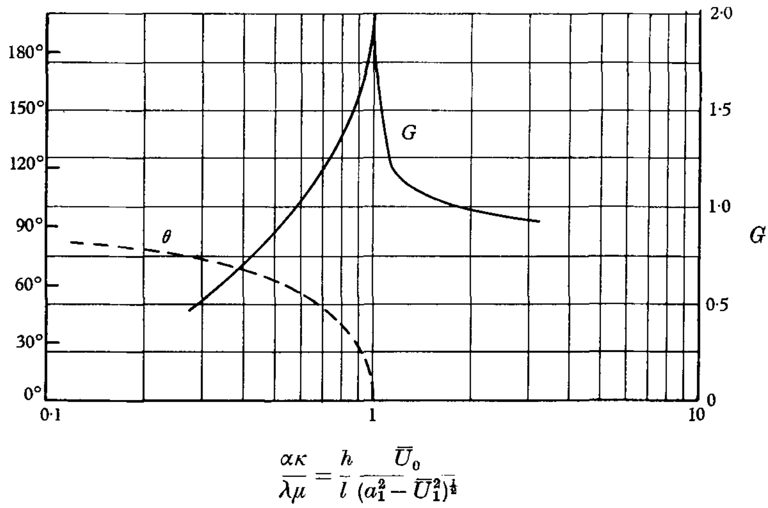

FIGURE 5. Phase θ and amplitude G of $\psi_t/|\delta U_0|$ *vs* $\alpha k/\lambda\mu$ for shock strength $\overline{P}_1/\overline{P}_0 = 1{\cdot}514$, $\alpha/\mu = 2{\cdot}085$.

where

$$G = \begin{cases} 1+\dfrac{C_\alpha}{\Pi_{21}a_0/a_1}\left[\dfrac{\alpha\kappa}{\lambda\mu}\left\{1-\left[1-\left(\dfrac{\lambda\mu}{\alpha\kappa}\right)^2\right]^{\frac{1}{2}}\right\}+\dfrac{\lambda\mu}{\alpha\kappa}\dfrac{M_1}{C\mu^2}\right] & (\alpha\kappa/\lambda\mu>1), \\ \left\{\left(1+\dfrac{C_\alpha}{\Pi_{21}a_0/a_1}\left[\dfrac{\alpha\kappa}{\lambda\mu}-\dfrac{\lambda\mu}{\alpha\kappa}\dfrac{M_1}{C\mu^2}\right]\right)^2+\left(\dfrac{C_\alpha}{\Pi_{21}a_0/a_1}\right)^2\left[1-\left(\dfrac{\alpha\kappa}{\lambda\mu}\right)^2\right]\right\}^{\frac{1}{2}} & (\alpha\kappa/\lambda\mu<1), \end{cases}$$

$$\theta = \begin{cases} 0 \quad (\alpha\kappa/\lambda\mu>1), \\ \tan^{-1}\left\{\dfrac{C_\alpha\left\{1-\left(\dfrac{\alpha\kappa}{\lambda\mu}\right)^2\right\}^{\frac{1}{2}}\Big/\Pi_{21}(a_0/a_1)}{1+\dfrac{C_\alpha}{\Pi_{21}(a_0/a_1)}\left[\dfrac{\alpha\kappa}{\lambda\mu}+\dfrac{\lambda\mu}{\alpha\kappa}\dfrac{M_1}{C\mu^2}\right]}\right\} \quad (\alpha\kappa/\lambda\mu<1). \end{cases}$$

The amplitude G and phase angle θ are plotted in figure 5 *vs* $\alpha\kappa/\lambda\mu$ for a shock strength of $\overline{P}_1/\overline{P}_0 = 1{\cdot}514$. At this shock strength the time variation of

$$p_{+\text{trans}}a_1/|u_0|\Pi_{21}a_0\sin\lambda y$$

is computed from equations (8.5) and (8.7) for large τ and from equation (9.10) for small τ. The results are plotted in figure 6 for $\alpha\kappa/\lambda\mu = \frac{1}{2}$ and 2·085, and in

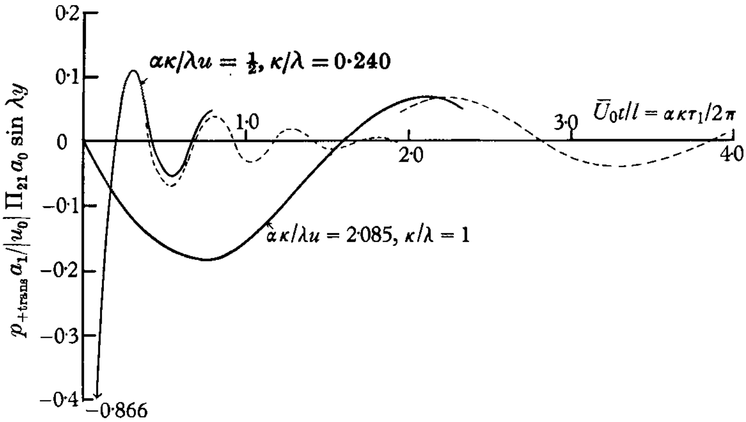

FIGURE 6. Transient component of pressure disturbance immediately behind shock *vs* time for $\alpha k/\lambda u = \frac{1}{2}$, $2{\cdot}085$; $\bar{P}_1/\bar{P}_0 = 1{\cdot}514$, $M_0 = 1{\cdot}2$. ———, Bessel function expansion; – – – –, asymptotic expression.

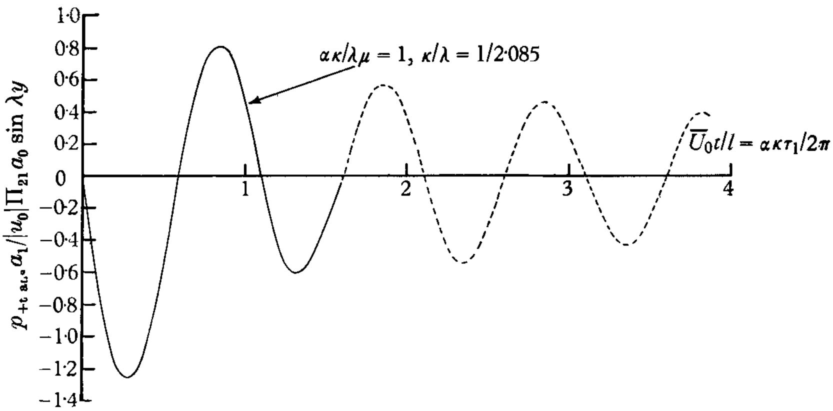

FIGURE 7. Transient component of pressure disturbance immediately behind shock *vs* time for $\alpha k/\lambda u = 1$, $\bar{P}_1/\bar{P}_0 = 1{\cdot}514$, $M_0 = 1{\cdot}2$. ———, Bessel function expansion; – – – –, asymptotic expression.

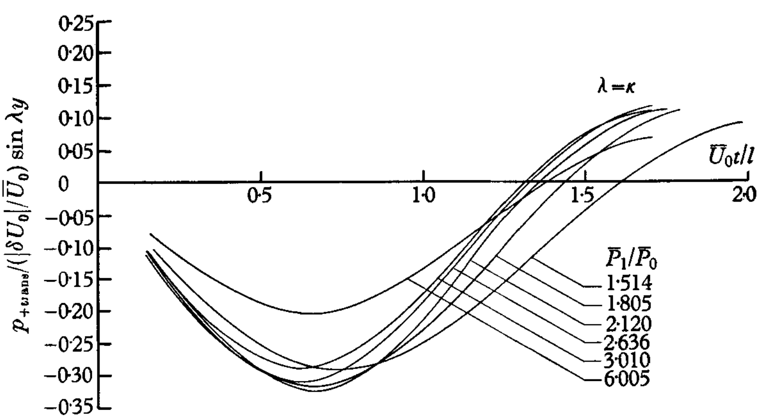

FIGURE 8. Transient component of pressure disturbance immediately behind shock for $\lambda\mu\tau \ll 1$ (initial behaviour), $\lambda = \kappa$, and different shock strengths.

figure 7 for $\alpha\kappa/\lambda\mu = 1$. The value of 2·085 was chosen since at this value $\lambda = \kappa$. It is interesting to note that when $\lambda = \kappa$ (i.e. square vortex cells), the maximum magnitude of $p_{+\text{trans}}$ is an order of magnitude greater than the amplitude of $p_{+\infty}$. In figure 8 the time variation of $p_{+\text{trans}}/(|\delta U_0|/\overline{U}_0)\sin\lambda y$ for $\lambda = \kappa$ at shock strength from 1·514 to 6·005 is plotted. Since $p_{+\text{trans}}$ is normalized with respect to $\overline{P}_1$ we can see from figure 8 that, in this range of shock strengths, the transient pressure component rises in magnitude roughly in proportion to the shock strength.

11. Shock displacement for a single column of vortices

The above solution can be used to solve a second problem which is of some interest, namely the convection through a shock of a single column of vortex cells parallel to the y-axis. For this case the upstream disturbance is specified by

$$\left.\begin{aligned} &u_0 = v_0 = 0 \quad (0 \geqslant \overline{U}_0 t - x, \quad \overline{U}_0 t - x \geqslant (\pi/\kappa) = \tfrac{1}{2}l); \\ &\left.\begin{aligned} u_0 &= |u_0| \sin\kappa(\overline{U}_0 t - x)\sin\lambda y \\ v_0 &= -|u_0|(\kappa/\lambda)\cos\kappa(\overline{U}_0 t - x)\cos\lambda y \end{aligned}\right\} \quad (\pi/\kappa) > \overline{U}_0 t - x > 0. \end{aligned}\right\} \tag{11.1}$$

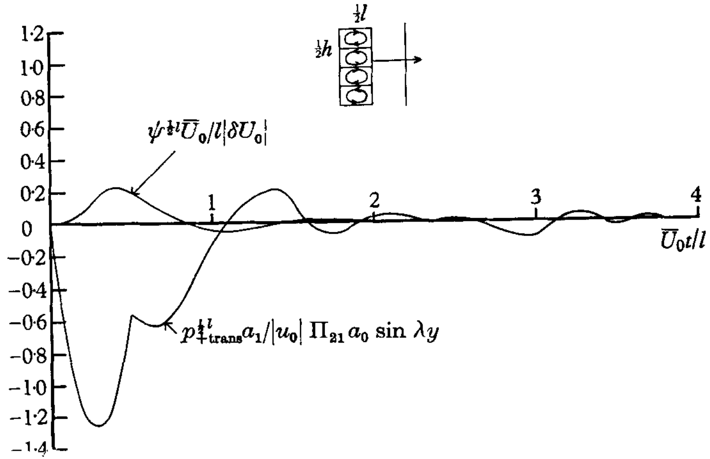

FIGURE 9. Shock displacement and pressure disturbance for a single column of vortices, with $\alpha\kappa/\lambda\mu = 1$, $\lambda/\kappa = 2{\cdot}085$, $\overline{P}_1/\overline{P}_0 = 1{\cdot}514$, $M_0 = 1{\cdot}2$.

Calling the shock displacement for this case $\psi^{\frac{1}{2}l}$ we have by simple superposition

$$\psi_t^{\frac{1}{2}l} = \psi_t(y,t) + \psi_t(y, t - l/2\overline{U}_0), \tag{11.2}$$

ψ_t being the displacement velocity found previously for the semi-infinite vortex field. Integration of equation (11.2) yields the displacement $\psi^{\frac{1}{2}l}$ which is plotted in figure 9 together with $p^{\frac{1}{2}l}_{+\text{trans}}a_1/|u_0|\Pi_{21}a_0\sin\lambda y$ as a function of time for $\alpha\kappa/\lambda\mu = 1$, $\overline{P}_1/\overline{P}_0 = 1{\cdot}514$.

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